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A Stochastic Return Map for Stochastic Differential Equations

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A method is presented for constructing a stochastic return map from a stochastic differential equation containing a locally stable limit cycle and small-amplitude $[O(\varepsilon)]$ additive Gaussian colored noise. The construction is valid provided the correlation time is $O(\varepsilon)$ or $O(1)$. The effective noise in the return map has nonzero $O(\varepsilon^2)$ mean and is state dependent. The method is applied to a model dynamical system, illustrating how the effective noise in the return map depends on both the original noise process and the local deterministic dynamics.

KEY WORDS: Stochastic map; stochastic differential equation; limit cycle; Gaussian colored noise.

1. INTRODUCTION

Stochastic differential equations (SDEs) have been used to describe a wide variety of systems in physics, chemistry, and biology.[1-5] The random character of solutions to such equations often represents unpredictability in the physical system due to the weak coupling of the system to its environment. Alternatively, randomness may represent uncertainty due to physical processes not explicitly included in the model. In both cases the characteristics of the noise process can only be determined by physical considerations. For example, if the noise source is independent of the state of the system, then additive noise is appropriate; otherwise, multiplicative or state-dependent noise is required. The characteristics of the noise process play an important role in determining the system's behavior. State-dependent noise, for example, introduces the possibility of nonequilibrium

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noise-induced transitions analogous to equilibrium phase transitions.[4,5] A qualitatively correct description of the dynamics of a physical system requires that the noise process be correctly chosen.

Many authors have studied the behavior of SDEs containing limit cycles, often in the context of population biology or solid-state physics.[6–13] One technique which has proved useful for studying deterministic systems with limit cycles is to construct a return map for the dynamics using a Poincaré surface of section.[14] In constructing a return map, continuous flow in the n-dimensional phase space of the system is reduced to discrete jumps in an $(n-1)$-dimensional oriented surface Σ: the successive properly oriented intersections of the phase space trajectory with Σ make up the discrete trajectory of the return map. Since the phase space trajectory of a system of SDEs is continuous, it is possible to construct a stochastic return map. The dynamics of noisy maps has been the object of many studies,[15–21] but the character of the noise included in these maps has so far been chosen in an ad hoc manner.

In this paper we address the question of how one determines the character of the noise in a stochastic return map. When the map results from the reduction of an SDE, the answer is that the noise process is determined from physical considerations at the level of the SDE, and then carried through the reduction process. We shall use a Poincaré surface of section to explicitly construct a stochastic return map from an SDE whose corresponding ordinary differential equation (ODE) contains a limit cycle. As we shall see, the effective noise which enters into the return map is significantly more complex than the simple noise process we put into the SDE.

One important aspect of SDEs with limit cycles is the phenomenon of phase diffusion.[21] A limit cycle $\Gamma = \{\gamma(t)\}$ of an autonomous dynamical system necessarily has a neutral eigenvalue in the direction tangent to Γ reflecting the fact that $\gamma(t+t_0)$ describes the same limit cycle. As a result, noise components tangent to the limit cycle cause perturbations which are not damped by the deterministic dynamics, and the system exhibits a random walk in phase. An ensemble of systems starting in phase on the limit cycle will thus diffuse to cover the entire limit cycle. From the viewpoint of an individual trajectory, the time for the system to return to a Poincaré surface is a stochastic quantity depending on the noise realization. An externally forced system containing a limit cycle will also have such a neutral eigenvalue and exhibit phase diffusion provided the limit cycle is not phase-locked.

In a study of the effects of a rapidly fluctuating environment on systems of interacting species, White[6] constructed an asymptotic expansion similar to that described below, but used the noise correlation time rather than the noise amplitude as a small parameter. Included in his work is consideration of a limit cycle in two dimensions. He finds that the variance of the phase grows linearly, and calculates the variance of the distance from the limit cycle. The linear growth in the variance of the phase is a manifestation of phase diffusion.

In Section 2 we explicitly construct the stochastic return map for a system of SDEs containing small-amplitude additive Gaussian colored noise. The corresponding ODEs contain a locally stable limit cycle with period T. If the system starts in the neighborhood of the limit cycle, it will remain there, on the average, for a very long time. Depending on the global structure of phase space, the system may eventually escape from the limit cycle and visit other attractors.[9–12,21] While the system is near the limit cycle it is possible to construct a locally valid stochastic return map by asymptotically expanding about the limit cycle in powers of the small noise amplitude ε. The effective noise entering into the return map depends explicitly on both the deterministic dynamics evaluated on the limit cycle and on the details of the original noise process. The success of this expansion requires that both the time-integrated amplitude and the instantaneous amplitude of the noise be small. The requirement that the instantaneous amplitude be small implies that the correlation time τ_c be at least $O(\varepsilon)$. Specifically, we consider here the two cases, $\tau_c = O(1)$ and $\tau_c = O(\varepsilon)$. The method used by White,[6] on the other hand, is valid in the limit of zero correlation time, where the instantaneous noise amplitude is large and the expansion used here breaks down.

We calculate the moments of the first-order effective noise in Section 3. The calculation is complicated by the fact that the first-order effective noise depends on the original noise process evaluated at stochastic times. This randomness in the argument of the noise arises from the stochastic time for the trajectory to return to the Poincaré surface of section. We show that although the original noise entering the SDE has zero mean, the effective noise in the return map has nonzero $O(\varepsilon^2)$ mean. In addition, for the case where the original noise has $O(1)$ correlation time, we derive explicit expressions for both the mean and mean-square of the first-order effective noise.

In Section 4 we apply the method to a model 2-dimensional dynamical system to explore the interactions between the original noise process and specific features in the deterministic dynamics. The dynamical system contains a limit cycle with unit radius and constant angular frequency, and is perturbed by small-amplitude isotropic additive Gaussian colored noise.

We note, following L. Arnold, that the stability of a limit cycle in an SDE may be characterized rigorously by the spectrum of Liapunov exponents.[22] These exponents are nonstochastic and have the usual properties of Liapunov exponents for deterministic systems. Although we do not pursue this approach here, we note that our setting is equivalent to supposing that all but one Liapunov exponent are negative, with the remaining one being zero.

2. CONSTRUCTION OF A STOCHASTIC RETURN MAP

Consider an n-dimensional set of SDEs with small-amplitude additive Gaussian colored noise:

$$\dot{x} = f(x) + \varepsilon\xi(t), \qquad x, \xi(t) \in \mathbb{R}^n, \qquad \varepsilon \ll 1 \tag{1}$$

In the following, Latin subscripts shall refer to components in $\mathbb{R}^n$, and the Einstein summation convention will be used. The Gaussian colored noise process is completely described by its first two moments:[23]

$$\langle \xi(t) \rangle = 0, \qquad \langle \xi_i(t_1)\, \xi_j(t_2) \rangle = \frac{D_{ij}}{\tau_c}\, e^{-|t_1 - t_2|/\tau_c} \tag{2}$$

The noise process depends on two parameters: the correlation time τ_c and the noise strength matrix D. The matrix D is referred to as the noise strength, as it is the time-integrated mean-square amplitude of the noise. The instantaneous mean-square amplitude is given by the matrix D/τ_c. In the limit $\tau_c \to 0$, Gaussian colored noise becomes Gaussian white noise with mean-square amplitude $2D$. As white noise is delta correlated, the instantaneous amplitude is infinite, and the time-integrated amplitude is the strength of the delta function. We shall consider noise processes in which the correlation time is finite, but no greater than $O(1)$.

Assume that the corresponding set of ODEs obtained when $\varepsilon = 0$ contains a locally stable limit cycle $\Gamma = \{\gamma(t)\}$ of period $T \sim O(1)$: $\gamma(t+T) = \gamma(t)$. The dynamics near Γ shall be described in terms of the distance y from a "tracking point" on Γ with phase t_0:

$$y(t) = x(t) - \gamma(t + t_0), \qquad y(t) \sim O(\varepsilon) \tag{3}$$

Due to phase diffusion, $x(t)$ will drift out of phase with $\gamma(t + t_0)$, causing $y(t)$ to become large even though $x(t)$ is still near Γ.[21] Here we avoid this problem by following $y(t)$ only until it next pierces the Poincaré surface in the appropriate direction, and choosing for the next iteration a new tracking point which once again starts out in phase with $x(t)$. These statements are made more precise below.

Equation (1) can be rewritten as

$$\dot{y}_i(t) = J_{ij}(t)\, y_j(t) + \varepsilon\xi_i(t) + O(\varepsilon^2) \tag{4}$$

where J is the Jacobian matrix

$$J_{ij}(t) = \left.\frac{\partial f_i(x)}{\partial x_j}\right|_{x=\gamma(t+t_0)} \tag{5}$$

Equation (4) can be transformed into an integral equation using the Green's matrix

$$G(t_1, t_2) = \begin{cases} \exp \int_{t_2}^{t_1} dt' J(t'), & t_1 \geqslant t_2 \\ 0, & t_1 < t_2 \end{cases} \tag{6}$$

yielding

$$y_i(t) = G_{ij}(t, 0)\, y_j(0) + \varepsilon \int_0^t dt'\, G_{ij}(t, t')\, \xi_j(t') + O(\varepsilon^2) \tag{7}$$

In the definition of G, Eq. (6), the time-ordered exponential is implicit. The linearized deterministic time-T map is given by $G(T, 0)$, which has one unit eigenvalue corresponding to motion tangent to Γ, and $n-1$ eigenvalues with magnitude less than one corresponding to contraction onto Γ.

We demand that the vector $y(t)$ remain small over one period. As the deterministic limit cycle is stable, growth in y transverse to Γ can only occur due to the cumulative effect of the noise. In the direction parallel to Γ, growth in y appears as a phase shift, and may be due to both the deterministic dynamics and the noise. Since $G(T, t) \sim O(1)$ and $y(0) \sim O(\varepsilon)$, the deterministic phase shift over one period is small. The cumulative effect of the noise will be small provided (sufficient but not necessary)

$$\int_0^T dt'\, G(T, t') \cdot \xi(t') \sim \int_0^T dt'\, \xi(t') \sim O(1) \tag{8}$$

The integrals in Eq. (8) are stochastic quantities. The question arises: what is the magnitude of a randomly fluctuating quantity? If the mean is zero, the magnitude is best estimated by the rms deviation of the quantity. If the mean is nonzero and larger than the rms deviation, then the magnitude is best estimated by the mean. The second integral in Eq. (8) is a random vector with zero mean; its size is thus best estimated by the sum of the rms deviations of its components:

$$\int_0^T dt\, \xi(t) \sim O\left(\left\langle \int_0^T dt_1 \int_0^T dt_2\, \xi_i(t_1)\, \xi_i(t_2) \right\rangle^{1/2}\right) \tag{9}$$

If G contains off-diagonal elements, then the rms magnitude of the first integral in Eq. (8) has contributions from $\xi_i \xi_j$, $i \neq j$. Interchanging the order of the ensemble average and the time integrals and using Eq. (2) results in

$$\left\langle \int_0^T dt_1 \int_0^T dt_2\, \xi_i(t_1)\, \xi_j(t_2) \right\rangle^{1/2} = [2D_{ij}T + 2D_{ij}\tau_c(e^{-T/\tau_c} - 1)]^{1/2}$$

$$\sim (D_{ij})^{1/2} \tag{10}$$

Since $D_{ij} = \tau_c \langle \xi_i(t)\, \xi_j(t) \rangle$, $|D_{ij}| \leqslant \frac{1}{2}(D_{ii} + D_{jj})$ (no sum on i, j); bounding the trace of D thus bounds every element D_{ij}. Hence, for correlation times $O(1)$ or smaller, the stochastic trajectory $y(t)$ will, over one period, remain near the tracking point provided $\mathrm{Tr}(D) \sim O(1)$ or smaller. In the following, we shall assume this is the case.

The Poincaré surface of section Σ_{t_0} is chosen to be the plane normal to Γ at $\gamma(t_0)$; each properly oriented intersection of the trajectory with Σ_{t_0} then occurs near $\gamma(t_0)$ with a stochastic time $s \sim T$ between intersections. The projection operator $P(t_0)$ takes vectors in $\mathbb{R}^n$ into Σ_{t_0}:

$$P_{ij}(t_0) = \delta_{ij} - v_i(t_0)\, v_j(t_0) \tag{11}$$

where $v(t_0)$ is the unit vector tangent to Γ at $\gamma(t_0)$:

$$v(t_0) = \frac{f(\gamma(t_0))}{|f(\gamma(t_0))|} \tag{12}$$

The geometry is shown in Fig. 1. We define the projection of G into Σ_{t_0}:

$$G^{\perp}(t_1, t_2) = P(t_0) \cdot G(t_1, t_2) \tag{13}$$

We are interested in parameter values where the deterministic system is not near a bifurcation. The projection of the linearized deterministic time-T map, $G^{\perp}(T, 0)$, then has $n-1$ eigenvalues with magnitude less than one. The vector $v(t_0)$, the eigenvector of $G(T, 0)$ with unit eigenvalue, lies in the null space of $G^{\perp}(T, 0)$.

Without loss of generality we may suppose that the initial condition $x(0)$ is in Σ_{t_0}; then the vector $x(0) - \gamma(t_0)$ is normal to $v(t_0)$:

$$[x(0) - \gamma(t_0)] \cdot v(t_0) = y(0) \cdot v(t_0) = 0 \tag{14}$$

Intersections of the trajectory with Σ_{t_0} occur at times $t = s$ such that $x(s) - \gamma(t_0)$ is normal to $v(t_0)$:

$$[x(s) - \gamma(t_0)] \cdot v(t_0) = [y(s) + \gamma(s + t_0) - \gamma(t_0)] \cdot v(t_0) = 0 \tag{15}$$

As mentioned above, $s \sim T$.

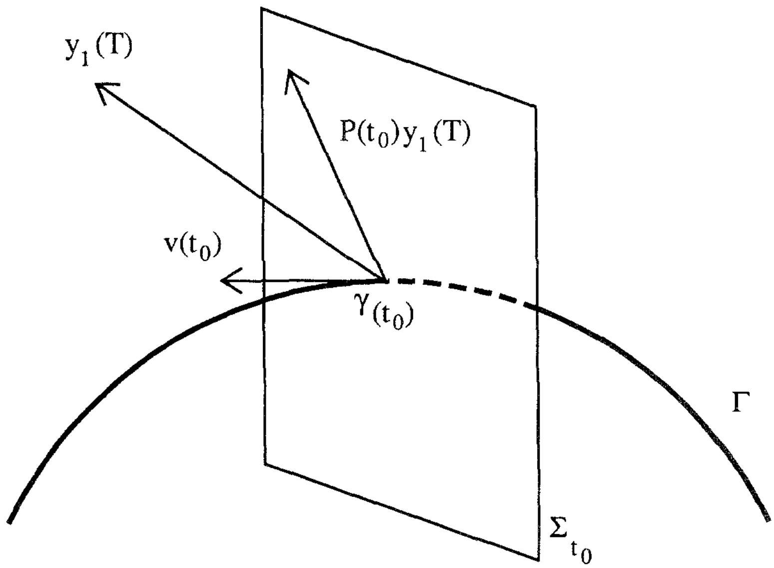

Fig. 1. Sketch of the surface of section Σ_{t_0} normal to the limit cycle Γ at $\gamma(t_0)$, the unit vector $v(t_0)$ tangent to Γ at $\gamma(t_0)$, and the effect of the projection operator $P(t_0)$ on the vector $y_1(T)$.

In order to construct the return map, we must expand both $y(t)$ and s in powers of ε. The lowest order part of the return map is independent of the size of the correlation time, while the powers of ε which appear in the higher order terms do depend on the size of τ_c. We shall thus focus on the lowest order part first. Expanding $y(t)$ and s,

$$\begin{aligned} y(t) &= \varepsilon y_1(t) + o(\varepsilon) \\ s &= s_0 + \varepsilon s_1 + o(\varepsilon) \end{aligned} \tag{16}$$

allows $y(s)$ to be written as

$$\begin{aligned} y(s) &= \varepsilon y_1(s_0) + o(\varepsilon) \\ &= \varepsilon G(s_0, 0) \cdot y_1(0) + \varepsilon \int_0^{s_0} dt\, G(s_0, t) \cdot \xi(t) + o(\varepsilon) \end{aligned} \tag{17}$$

and $\gamma(s + t_0)$ as

$$\gamma(s + t_0) = \gamma(s_0 + t_0) + \varepsilon s_1 f(\gamma(s_0 + t_0)) + o(\varepsilon) \tag{18}$$

The intersection condition (15) can be solved at $O(1)$ and $O(\varepsilon)$ to obtain

$$s_0 = T$$
$$s_1 = -\frac{v(t_0) \cdot y_1(T)}{|f(\gamma(t_0))|} \tag{19}$$

Thus, s_1, the first correction to the return time $s_0 = T$, is the first-order distance out of the plane Σ_{t_0} after one period divided by the velocity at t_0.

Let $z(m)$ represent the vector in Σ_{t_0} corresponding to a particular intersection with the correct orientation. Then the return map is of the form

$$z(m+1) = F(z(m)) + \zeta(m), \qquad z(m), \zeta(m) \in \mathbb{R}^{n-1} \tag{20}$$

where F is a deterministic function of z, and ζ is the effective noise. Let the time of the mth intersection be denoted by $\tau(m)$, and let the time between intersections be

$$s(m) = \tau(m+1) - \tau(m) \tag{21}$$

Note that $\tau(m)$ is a stochastic quantity. To calculate F and ζ, we set $y(0) = z(m)$ and use the above formalism to calculate

$$z(m+1) = y(s(m)) + \gamma(s(m) + t_0) - \gamma(t_0) \tag{22}$$

From the definition of $s(m)$ in (21) and the fact that $z(m) \in \Sigma_{t_0}$, $z(m+1) \in \Sigma_{t_0}$. Thus, at each iteration of the map we reinitialize y by choosing $\gamma(t_0)$ as the new tracking point, and then follow y once around Γ. Care must be taken to evaluate the noise process at the actual time, $\tau(m) + t$, rather than at the time since the previous intersection with Σ_{t_0}. Expanding z in powers of ε,

$$z(m) = \varepsilon z_1(m) + o(\varepsilon) \tag{23}$$

gives the first-order return map:

$$z_1(m+1) = P(t_0) \cdot y_1(T) \tag{24}$$

The deterministic part $F_1(z(m))$ is thus

$$F_1(z(m)) = G^{\perp}(T, 0) \cdot z_1(m) \tag{25}$$

while the first-order effective noise ζ_1 is

$$\zeta_1(m) = \int_0^T dt\, G^{\perp}(T, t) \cdot \xi(\tau(m) + t) \tag{26}$$

Due to the nature of the asymptotic expansion, F_1 and ζ_1 are linear in z_1 and ξ, respectively; higher order terms in F and ζ will be nonlinear. Since the deterministic system is away from a bifurcation, Hartman's theorem[14] guarantees the existence of a coordinate transformation linearizing the deterministic portion of the map F in some neighborhood of the fixed point $z=0$ $[x=\gamma(t_0)]$.

The first-order effective noise is independent of the present state, $z(m)$, and depends explicitly on the original noise process over times $\tau(m)$ to $\tau(m)+T$. Through $\tau(m)$, which is itself a stochastic quantity, ζ_1 depends on the the entire history of the original noise process. The first-order effective noise is studied in some detail in the next section.

To calculate higher order terms in the return map, one must extend the expansions (16) to include higher powers in ε. As we shall see below, the powers of ε which appear in the asymptotic expansions are determined by consistency considerations, and differ for correlation times with different magnitudes. We shall thus write the higher order terms using exponents $1<p_1<p_2\cdots$. The expansion for the intersection time s is

$$s=T+\varepsilon s_1+\varepsilon^{p_1}s_{p_1}+\varepsilon^{p_2}s_{p_2}+\cdots \tag{27}$$

The expansion for $y(s)$ is constructed from Eq. (7). Expanding around T in both the Green's matrix and the upper bound of the integrals, we obtain

$$y(s)=y(T)+(s-T)\,J(0)\cdot y(T)+\varepsilon\int_T^s dt\,\xi(t)+O(\varepsilon^2) \tag{28}$$

The first term in (28) arises from those terms in the expansion of (7) in which s is replaced everywhere by T. The second term in (28) results from the first term in the Taylor expansion of $G(s,0)$ and $G(s,t')$ around $s=T$, with the upper bound on the integrals being T. The third term is the lowest order piece of the integrals from T to s. The Green's matrix does not appear in the third term because t is close to s and the lowest order piece of $G(s,t)$ is the identity matrix. Using the expansion of s, Eq. (27), $y(s)$ becomes

$$\begin{aligned} y(s)=&\;y(T)+\varepsilon s_1\,J(0)\cdot y(T)+\varepsilon^{p_1}s_{p_1}J(0)\cdot y(T)\\ &+\varepsilon\int_T^{T+\varepsilon s_1} dt\,\xi(t)+\varepsilon\int_{T+\varepsilon s_1}^{T+\varepsilon s_1+\varepsilon^{p_1}s_{p_1}} dt\,\xi(t)+\cdots \end{aligned} \tag{29}$$

The integrals appearing in Eq. (29) have stochastic integrands and, since s is stochastic, stochastic bounds. To check on the consistency of the asymptotic expansion, we must know the magnitude of these integrals, which requires knowledge of their moments. In general, calculating the moments of these integrals is difficult. In the Appendix we calculate the moments of the first integral in Eq. (29) and show that its magnitude is the same as that of a similar integral with deterministic bounds; we henceforth assume that the other integrals with stochastic bounds which also arise have the same order of magnitude as similar integrals with deterministic bounds. These integrals take the form

$$I=\int_{t}^{t+\varepsilon^q s_q} dt'\,\xi(t') \tag{30}$$

where both t and s_q may be stochastic and $s_q \sim O(1)$. Our assumption is that

$$I\sim O\left(\int_0^{\varepsilon^q} dt'\,\xi(t')\right) \tag{31}$$

Since the mean of the integral on the right is zero, its magnitude is estimated by the rms deviation, which can be calculated in the same manner as Eq. (10). The result is

$$\left\langle \int_0^{\varepsilon^q} dt_1 \int_0^{\varepsilon^q} dt_2\, \xi_i(t_1)\,\xi_j(t_2)\right\rangle^{1/2} = [2D_{ij}\varepsilon^q + 2D_{ij}\tau_c(e^{-\varepsilon^q/\tau_c}-1)]^{1/2} \tag{32}$$

If $\tau_c \sim O(\varepsilon^p)$, then using the Taylor expansion of the exponential in (32), the fact that $\mathrm{Tr}(D)\sim O(1)$, and the assumption (31), we obtain

$$I\sim\begin{cases} O(\varepsilon^{q/2}), & p>q \\ O(\varepsilon^{q-p/2}), & p<q \end{cases} \tag{33}$$

When $p=q$, both expressions in (33) reduce to $O(\varepsilon^{q/2})$. The first case, $p>q$, is just the familiar fact that the rms diffusion distance goes like the square root of the time when the correlation time is much shorter than the diffusion time.

To satisfy the intersection condition (15), any term of order ε^p appearing in $y(s)$ must be canceled by a term of the same order in $\gamma(s+t_0)$. A term of order ε^p will only appear in the Taylor expansion of $\gamma(s+t_0)$ if the asymptotic expansion of s has such a term. Through the integrals discussed above, terms in the expansion of s generate terms in the expansion of $y(s)$. This circularity gives rise to a consistency requirement, in which all terms generated by any term in the asymptotic expansion of $y(s)$ must be the same order as terms already present in the expansion.

When $\tau_c \sim O(1)$, integrals of the noise process over times of $O(\varepsilon^q)$, $q \geqslant 0$, are themselves $O(\varepsilon^q)$, and the expansions for y and s contain only integer powers of ε. If $\tau_c \sim O(\varepsilon)$, however, integrals of the noise process over $O(\varepsilon)$ times are $O(\varepsilon^{1/2})$, and the fourth term in Eq. (29) is $O(\varepsilon^{3/2})$. Consistency then requires an $\varepsilon^{3/2}$ term in the expansion of s, resulting in a term in the expansion of y containing an integral of the noise process over an $O(\varepsilon^{3/2})$ time. This integral is $O(\varepsilon)$. In this way, one finds that the expansions for the case $\tau_c \sim O(\varepsilon)$ are consistent if they contain half-integer powers of ε. The form of the asymptotic expansions thus depends on the magnitude of τ_c.

Once the form of the expansions is settled, it is straightforward to solve the intersection condition (15) order by order, resulting in explicit expressions for s_{p_1}, s_{p_2}, etc. Higher order corrections to the stochastic return map can then be obtained in the manner described above for the first-order map: set $y(0) = z(m)$ and use Eq. (22) to calculate $z(m+1)$. For both $\tau_c \sim O(1)$ and $\tau_c \sim O(\varepsilon)$, ζ_2 [the coefficient of the $O(\varepsilon^2)$ part of the effective noise] depends quadratically on z_1 and ξ, and is thus state dependent and has nonzero mean.[24]

When $\tau_c \sim O(\varepsilon^2)$ or smaller, the situation is quite different. As is the case when $\tau_c \sim O(\varepsilon)$, the integral of the noise process over an order-ε time is $O(\varepsilon^{1/2})$, which generates an order-$\varepsilon^{3/2}$ term in the expansion of s. Now, however, $\varepsilon^{3/2} \gg \tau_c$ and the $O(\varepsilon^{3/2})$ term generates a term of order $\varepsilon^{7/4}$, which is still much greater than τ_c. This process continues *ad infinitum*, and the asymptotic expansion breaks down. We may think of this process of generating new terms as a mapping: terms of order $\varepsilon^q \gg \tau_c$ generate new terms of order $\varepsilon^{q'}$, where $q' = 1 + q/2$. This map has a stable fixed point at $q = 2$; the generation of an infinite number of terms with new powers of ε corresponds to the infinite number of iterations required to reach the fixed point from an initial condition at $q = 1$. The above formalism for constructing a stochastic return map thus breaks down when $\tau_c \sim O(\varepsilon^2)$.

We required earlier that the integrated amplitude of the noise term $\varepsilon\xi$ be small, i.e., $\mathrm{Tr}(D) \sim O(1)$ or smaller. Together with the new requirement that $\tau_c \sim O(\varepsilon)$ or larger, we obtain a restriction on the instantaneous rms amplitude of ξ,

$$\langle \xi_i(t)\, \xi_j(t) \rangle^{1/2} = (D_{ij}/\tau_c)^{1/2} \sim O(\varepsilon^{-1/2}) \quad \text{or smaller} \tag{34}$$

Since the noise entering into the SDE is $\varepsilon\xi$, (34) is equivalent to requiring that the instantaneous rms amplitude of the noise term in the SDE be small, i.e., $O(\varepsilon^{1/2})$ or smaller. Thus, the success of the technique presented above requires that both the instantaneous and the time-integrated rms amplitudes of the noise term in the SDE be small.

3. THE FIRST-ORDER EFFECTIVE NOISE

In this section we examine the first two moments of the first-order effective noise. From the definition of ζ_1, Eq. (26), we see that the moments of the first-order effective noise depend on the moments of $\xi(\tau(m)+t)$. When $\tau_c \sim O(1)$ the only small parameter is ε and the lowest order contribution to the moments of $\xi(\tau(m)+t)$ can be calculated in closed form. This is not the case when $\tau_c \sim O(\varepsilon)$, when two small parameters are present.

We first assume $\tau_c \sim O(1)$. The chief difficulty in evaluating the moments of $\xi(\tau(m)+t)$ is that $\tau(m)$ is itself a stochastic variable:

$$\tau(m) = mT + \varepsilon(s_1(0) + \cdots + s_1(m-1)) + O(\varepsilon^2) \tag{35}$$

where the stochasticity arises through $s_1(m)$,

$$s_1(m) = -\frac{v(t_0)}{|f(\gamma(t_0))|} \cdot \left[G(T,0) \cdot z_1(m) + \int_0^T dt\, G(T,t) \cdot \xi(\tau(m)+t) \right] \tag{36}$$

which, using Eq. (25) and (26), can be written as

$$\begin{aligned} s_1(m) = -\frac{v(t_0)}{|f(\gamma(t_0))|} \cdot \Bigg[& G(T,0) \cdot [G^{\perp}(T,0)]^m \cdot z_1(0) \\ & + \int_0^T dt\, G(T,t) \cdot \xi(\tau(m)+t) + \sum_{n=0}^{m-1} G(T,0) \cdot [G^{\perp}(T,0)]^n \\ & \cdot \int_0^T dt\, G^{\perp}(T,t) \cdot \xi(\tau(m-n-1)+t) \Bigg] \end{aligned} \tag{37}$$

With the aid of the delta function, $\tau(m)$ can be removed from the argument of ξ. Fourier transform of the delta function and Taylor series expansion of the resulting exponential allow $\xi(\tau(m)+t)$ to be expressed as a polynomial in ξ and $\tau(m)$:

$$\begin{aligned} \xi(\tau(m)+t) &= \int_{-\infty}^{\infty} dt'\, \xi(t')\, \delta(\tau(m)+t-t') \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} d\omega\, e^{i\omega(t-t')} \sum_{\alpha=0}^{\infty} \frac{(i\omega)^\alpha}{\alpha!}\, \xi(t')(\tau(m))^\alpha \end{aligned} \tag{38}$$

We calculate the mean square of ξ first. Using Eq. (38) and (35), one obtains

$$\langle \xi(\tau(m)+t_1)\,\xi(\tau(n)+t_2)\rangle$$
$$=\frac{1}{4\pi^2}\int_{-\infty}^{\infty} dt_3 \int_{-\infty}^{\infty} dt_4 \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2\, e^{i\omega_1(t_1-t_3)}e^{i\omega_2(t_2-t_4)}$$
$$\times \sum_{\alpha,\beta=0}^{\infty} \frac{(i\omega_1)^\alpha\,(i\omega_2)^\beta}{\alpha!\,\beta!}\langle \xi(t_3)\,\xi(t_4)\rangle(mT)^\alpha\,(nT)^\beta + O(\varepsilon) \tag{39}$$

The average is now evaluated, and the above expansion into integrals and sums is easily reversed. The sums give the exponentials $e^{i\omega_1 mT}e^{i\omega_2 nT}$, and the ω integrals are once again delta functions. The result is

$$\langle \xi(\tau(m)+t_1)\,\xi(\tau(n)+t_2)\rangle=\frac{D}{\tau_c}\,e^{-|t_1-t_2+(m-n)T|/\tau_c}+O(\varepsilon) \tag{40}$$

The second moment of the effective noise is thus

$$\langle \zeta_{1i}(m)\,\zeta_{1j}(n)\rangle=\int_0^T dt_1 \int_0^T dt_2\, G^{\perp}_{ik}(T,t_1)\,G^{\perp}_{jl}(T,t_2)$$
$$\times\frac{D_{kl}}{\tau_c}\,e^{-|t_1-t_2+(m-n)T|/\tau_c}+O(\varepsilon) \tag{41}$$

The mean is calculated in a similar manner. From Eq. (38) we see that the mean of $\xi(\tau(m)+t)$ requires the average $\langle \xi(t')(\tau(m))^\alpha\rangle$. Since $\tau(m)$ is deterministic at first order, the lowest order contribution comes from a single s_1 and is $O(\varepsilon)$:

$$\langle \xi(t')(\tau(m))^\alpha\rangle=\varepsilon\alpha(mT)^{\alpha-1}\,S(m,t')+O(\varepsilon^2) \tag{42}$$

where $S(m, t')$ is the vector

$$S_i(m,t')\equiv\sum_{n=0}^{m-1}\langle \xi_i(t')\,s_1(n)\rangle$$
$$=\frac{-v_j(t_0)}{|f(\gamma(t_0))|}\sum_{n=0}^{m-1}\left[\int_0^T dt''\,G_{jk}(T,t'')\,\frac{D_{ik}}{\tau_c}\,e^{-|t'-t''-nT|/\tau_c}\right.$$
$$+G_{jk}(T,0)\sum_{p=0}^{n-1}[G^{\perp}(T,0)]^p_{kl}$$
$$\left.\times\int_0^T dt''\,G^{\perp}_{lm}(T,t'')\,\frac{D_{im}}{\tau_c}\,e^{-|t'-t''-(n-p-1)T|/\tau_c}\right] \tag{43}$$

The sum over α from Eq. (38) may be carried out, resulting in $i\omega e^{i\omega mT}$. The factor of $i\omega$ is now rewritten as a time derivative and the remaining ω integral is seen to be the delta function. The mean of $\xi(\tau(m)+t)$ is thus

$$\langle \xi(\tau(m)+t) \rangle = \varepsilon \frac{d}{dt} S(m, t+mT) + O(\varepsilon^2) \tag{44}$$

The average of the first-order effective noise is

$$\langle \zeta_1(m) \rangle = \varepsilon \int_0^T dt\, G^{\perp}(T, t) \cdot \frac{d}{dt} S(m, t+mT) + O(\varepsilon^2) \tag{45}$$

and is $O(\varepsilon)$. Thus, an $O(\varepsilon)$ zero mean noise process in an SDE is transformed under the construction of a return map into an $O(\varepsilon)$ effective noise process with a nonzero $O(\varepsilon^2)$ mean and an $O(\varepsilon)$ rms amplitude.

We now consider the case $\tau_c \sim O(\varepsilon)$. The term given by Eq. (40) is still present, but is now $O(\varepsilon^{-1})$. In addition, there are terms which depend on the mean square of the noise through $\langle \xi\xi s_1 s_1 \rangle$. These terms have an explicit factor of ε^2, and were previously ignored as being higher order. Now, however, the dependence on τ_c can cancel the explicit factor of ε^2. If one assumes that these terms are higher order, one can solve for them explicitly by iterating the assumed lowest order solution, Eq. (40). One finds that there are terms which contain factors of $1/\tau_c^3$, which do indeed cancel the ε^2, rendering the iteration scheme invalid. In this case we cannot express the lowest order part of the mean square of $\xi(\tau(m)+t)$ in closed form. As in the previous case, the mean depends on the mean square; thus, the mean also cannot be written in closed form. The term given by Eq. (45) will still contribute to the mean, however, and is $O(\varepsilon)$. Thus, the mean first-order effective noise is at least $O(\varepsilon)$.

4. A MODEL DYNAMICAL SYSTEM

We now apply the method described in the previous section to a model dynamical system. To first order, the only information needed about the deterministic dynamics is the Green's matrix $G(t, t')$ obtained by solving the linearized ODE. At successively higher order one needs to know the Jacobian matrix $J(t)$, then the tensor of second derivatives, etc., all of which can be calculated directly from the velocity vector field $f(x)$. One of the simplest dynamical systems containing a limit cycle is a two-dimensional set of ODEs which decouple into radial and angular components. The model considered below differs from this simple case in that the angular frequency depends on the radius, causing deviations from solid-body rotation. While the model is most intuitively defined in polar coordinates, the calculation is carried out in Cartesian coordinates, where the noise process is defined. The noise process in the SDE is isotropic additive Gaussian colored noise:

$$\langle \xi(t) \rangle = 0, \qquad \langle \xi_i(t)\, \xi_j(t') \rangle = (\delta_{ij}/\tau_c) \exp(-|t-t'|/\tau_c) \tag{46}$$

In polar coordinates the deterministic system is

$$\begin{aligned} \dot{r} &= r - r^{1-\lambda}, \qquad \lambda < 0 \\ \dot{\theta} &= \omega_0 + \omega(r), \qquad \omega(1) = 0 \end{aligned} \tag{47}$$

which contains a limit cycle with unit radius and constant angular velocity $\omega_0 = 2\pi/T$. In Cartesian coordinates the limit cycle is

$$\Gamma = \left\{ \begin{pmatrix} \gamma_x(t) \\ \gamma_y(t) \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \cos \omega_0 t \\ \sin \omega_0 t \end{pmatrix} \right\} \tag{48}$$

and the unit vector tangent to Γ is

$$v(t) = \begin{pmatrix} -\gamma_y(t) \\ \gamma_x(t) \end{pmatrix} \tag{49}$$

The exponent λ is the radial eigenvalue and is negative, as Γ is assumed to be locally stable. Since Eq. (47) has circular symmetry, t_0 may be set to zero. The Green's matrix is most easily found by integrating the linearized form of Eq. (47) and then transforming to Cartesian coordinates, resulting in

$$\begin{aligned} G_{ij}(t, t') &= e^{\lambda(t-t')} \gamma_i(t)\, \gamma_j(t') + v_i(t)\, v_j(t') \\ &\quad + \frac{\omega'(1)}{\lambda} (e^{\lambda(t-t')} - 1)\, v_i(t)\, \gamma_j(t') \end{aligned} \tag{50}$$

From Eq. (26) the first-order effective noise is

$$\zeta_{1i}(m) = \int_0^T dt\, e^{\lambda(T-t)} \gamma_i(0)\, \gamma_j(t)\, \xi_j(\tau(m) + t) \tag{51}$$

It is straightforward to use the results of the previous section to calculate the lowest order parts of the first two moments of ζ_1 for the case $\tau_c \sim O(1)$. The mean $\langle \zeta_1(m) \rangle$ depends explicitly on m, while the mean square $\langle \zeta_1(m)\, \zeta_1(n) \rangle$ depends only on the difference $|m-n|$. We shall examine the equilibrium behavior of the mean, i.e., the limit of large m.

While this restriction is not necessary for the calculation, it simplifies the resulting formulas considerably. The mean of the first-order effective noise is then

$$\lim_{m \to \infty} \langle \zeta_1(m) \rangle = \varepsilon\gamma(0)\, C(e^{-T/\tau_c} - e^{\lambda T}) \tag{52}$$

where

$$C = \frac{1}{(1+\lambda\tau_c)^2 + \omega_0^2\tau_c^2}\left[\frac{\omega'(1)}{(1-\lambda\tau_c)^2 + \omega_0^2\tau_c^2}\,\frac{1-\lambda^2\tau_c^2+\omega_0^2\tau_c^2}{\omega_0\lambda} - \frac{1}{1+\omega_0^2\tau_c^2}\left(\omega'(1)\,\frac{1+\lambda\tau_c+\omega_0^2\tau_c^2}{\omega_0\lambda} - \lambda\tau_c^2\right)\right] \tag{53}$$

It is interesting to look at the factors which determine the sign of the mean. When the local rotation rate is independent of the radius, $\omega'(1) = 0$, the sign of $\langle \zeta_1 \rangle$ is identical to the sign of $1/\tau_c + \lambda$. Thus, for $\omega'(1) = 0$, the effective noise tends to push the system to the interior (exterior) of Γ when the correlation time of the original noise process is larger (smaller) than the deterministic decay time for contraction onto Γ. Further, for fixed λ and τ_c, there is a value of $\omega'(1)$ for which the sign of $\langle \zeta_1 \rangle$ changes to the opposite of that at $\omega'(1) = 0$. This value is determined by solving $C = 0$ for $\omega'(1)$, and depends on ω_0, λ, and τ_c in a complex manner. For all $\omega'(1)$, the lowest order contribution to the mean effective noise is zero when $1/\tau_c = |\lambda|$.

The covariance of the effective noise is

$$\langle \zeta_{1i}(m)\, \zeta_{1j}(m) \rangle = \frac{\gamma_i(0)\,\gamma_j(0)}{(1-\lambda^2\tau_c^2)^2 + 2\omega_0^2\tau_c^2(1+\lambda^2\tau_c^2) + \omega_0^4\tau_c^4} \times \left[\frac{1-\lambda^2\tau_c^2+\omega_0^2\tau_c^2}{\lambda}(e^{2\lambda T} - 1) + \tau_c(1-\lambda^2\tau_c^2 - \omega_0^2\tau_c^2)(2e^{(\lambda - 1/\tau_c)T} - e^{2\lambda T} - 1)\right] \tag{54}$$

and is independent of m. When $m \neq n$ the second moment is

$$\langle \zeta_{1i}(m)\, \zeta_{1j}(n) \rangle = e^{-|m-n|\,T/\tau_c}\gamma_i(0)\,\gamma_j(0) \times \frac{\tau_c(1-\lambda^2\tau_c^2 - \omega_0^2\tau_c^2)}{(1-\lambda^2\tau_c^2)^2 + 2\omega_0^2\tau_c^2(1+\lambda^2\tau_c^2) + \omega_0^4\tau_c^4} \times (e^{(\lambda + 1/\tau_c)T} + e^{(\lambda - 1/\tau_c)T} - e^{2\lambda T} - 1) \tag{55}$$

and depends on m and n only through their difference $|m-n|$. Thus, unlike the mean, the second moment of the effective noise is stationary. Another difference from the mean is that the second moment is independent of $\omega'(1)$. Notice that the limit $n \to m$ of Eq. (55) is not Eq. (54). The difference is due to the absolute value in the exponent of Eq. (41), and the restriction that m, n take integer values. When $m = n$, $t_1 - t_2 + (m-n)T$ changes sign along the line $t_1 = t_2$, which is in the region of integration. The integration must be broken into two regions, $t_1 > t_2$ and $t_1 < t_2$, and one obtains boundary terms on $t_1 = t_2$. When $m \neq n$, $t_1 - t_2 + (m-n)T$ is nonzero throughout the region of integration and no such boundary terms appear.

5. CONCLUSIONS

For an autonomous SDE with small-amplitude additive Gaussian colored noise containing a locally stable limit cycle it is possible to calculate a local stochastic return map provided the correlation time of the noise is $O(\varepsilon)$ or $O(1)$. The effective noise in the resulting return map has nonzero mean and is state dependent. As seen in a simple model dynamical system, the properties of the effective noise depend in a complex manner on both the local deterministic dynamics and the details of the original noise process. Even in this simple dynamical system the effective noise has non-zero mean and the system tends to be pushed away from the deterministic limit cycle, a tendency which is balanced by the deterministic contraction.

Although the return map is not the only map that may be constructed,[21] we believe that for processes driven by colored noise it is the most useful. We note, however, that when the noise at the SDE level is Gaussian white noise, the time-T map, appropriately defined, also has a number of useful properties.[25,26] Since colored noise may be thought of as the output of a Langevin equation with Gaussian white noise, the latter approach is available at the cost of increasing the dimension of the system.

APPENDIX. MAGNITUDE OF AN INTEGRAL WITH A STOCHASTIC BOUND

The first integral in Eq. (29), to be called I, is a stochastic vector with stochastic variables in both the upper bound and the integrand:

$$I_i \equiv \int_T^{T+\varepsilon s_1} dt\, \xi_i(t) = \int_0^{\varepsilon s_1} dt\, \xi_i(T+t) \tag{56}$$

With the help of the Heaviside step function $\Theta(t)$ and its integral representation

$$\Theta(t) = \frac{1}{2\pi i} \int_C d\omega\, \frac{e^{i\omega t}}{\omega} \tag{57}$$

we transfer the stochastic upper bound to the integrand. Here the contour C is the real ω axis from $-\infty$ to ∞, except at $\omega=0$, where C goes around the pole with $\text{Im}(\omega)<0$. There are two terms I_+ and I_-, as s_1 may be either positive or negative:

$$\begin{aligned} I &= \sum_{\sigma=\pm 1} \int_0^{\sigma\infty} dt\, \xi(T+t)\, \Theta(\sigma(\varepsilon s_1 - t)) \\ &\equiv \sum_{\sigma=\pm 1} I_\sigma \end{aligned} \tag{58}$$

The first-order correction to the intersection time s_1 has a deterministic part $\bar{s}_1$ and a stochastic part s_1' given by Eq. (19). Substituting the integral representation of $\Theta(t)$ and expanding the exponential of s_1' results in

$$I_\sigma = \frac{1}{2\pi i} \int_0^{\sigma\infty} dt \int_C d\omega \frac{e^{i\omega\sigma(\varepsilon\bar{s}_1 - t)}}{\omega} \sum_{n=0}^{\infty} \frac{(i\varepsilon\omega\sigma)^n}{n!} (s_1')^n\, \xi(T+t) \tag{59}$$

The average of I_σ now depends on the average of $(s_1')^n\, \xi(T+t)$:

$$\langle (s_1')^n\, \xi_i(T+t)\rangle = \begin{cases} n!!\, A^{(n-1)/2} B_i(t), & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \tag{60}$$

where

$$\begin{aligned} A &\equiv \langle (s_1')^2 \rangle \\ &= \int_0^T dt_1 \int_0^T dt_2 \frac{v_i(t_0)\, v_j(t_0)}{|f(\gamma(t_0))|^2} G_{ik}(T, t_1)\, G_{jl}(T, t_2) \frac{D_{kl}}{\tau_c} e^{-|t_1 - t_2|/\tau_c} \end{aligned} \tag{61}$$

and

$$\begin{aligned} B_i(t) &\equiv \langle s_1' \xi_i(T+t) \rangle \\ &= \int_0^T dt_1 \frac{-v_j(t_0)}{|f(\gamma(t_0))|} G_{jk}(T, t_1) \frac{D_{ik}}{\tau_c} e^{-|t_1 - T - t|/\tau_c} \end{aligned} \tag{62}$$

For n odd, $n=2m+1$, $n!!/n! = (1/2^m m!)$, and we obtain

$$\begin{aligned} \langle I_\sigma \rangle &= \frac{1}{2\pi i} \int_0^{\sigma\infty} dt \int_C d\omega \frac{e^{i\omega\sigma(\varepsilon\bar{s}_1 - t)}}{\omega} B(t) \sum_{m=0}^{\infty} \frac{(i\varepsilon\omega\sigma)^{2m+1}}{2^m m!} A^m \\ &= \frac{\varepsilon\sigma}{2\pi} \int_0^{\sigma\infty} dt \int_C d\omega\, e^{i\omega\sigma(\varepsilon\bar{s}_1 - t)} e^{-\varepsilon^2\omega^2 A/2} B(t) \end{aligned} \tag{63}$$

There is no longer a pole in the ω integral and the integral over C becomes the Fourier transform, yielding

$$\langle I_\sigma \rangle = \frac{\sigma}{(2\pi A)^{1/2}} \int_0^{\sigma\infty} dt\, B(t) e^{-(\varepsilon \bar{s}_1 - t)^2/2\varepsilon^2 A} \tag{64}$$

Finally, summing over σ gives the average of I:

$$\langle I_i \rangle = \frac{1}{(2\pi A)^{1/2}} \int_{-\infty}^{\infty} dt\, B_i(t) e^{-(\varepsilon \bar{s}_1 - t)^2/2\varepsilon^2 A} \tag{65}$$

A similar calculation gives the mean square of I:

$$\begin{aligned}\langle I_i I_j \rangle = {} & \frac{2D_{ij}}{(2\pi\varepsilon^2 A)^{1/2}} \int_{-\infty}^{\infty} dt[|t| + \tau_c(e^{-|t|/\tau_c} - 1)] e^{-(\varepsilon \bar{s}_1 - t)^2/2\varepsilon^2 A} \\ & + \frac{1}{A(2\pi\varepsilon^2 A)^{1/2}} \int_{-\infty}^{\infty} dt(t - \varepsilon \bar{s}_1) e^{-(\varepsilon \bar{s}_1 - t)^2/2\varepsilon^2 A} \\ & \times \int_0^t dt'[B_i(t)\, B_j(t') + B_j(t)\, B_i(t')]\end{aligned} \tag{66}$$

We can now estimate the size of I by examining the equations for the average and mean square of I, Eq. (65) and (66), respectively. The constant A, defined in Eq. (61), is, apart from the $O(1)$ vector $v/|f|$, the mean square of the first integral in Eq. (8). As this integral is $O(1)$, $A \sim O(1)$. The integrals in $\langle I \rangle$, $\langle I^2 \rangle$ ranging from negative to positive infinity contain Gaussians with exponent $-(\varepsilon \bar{s}_1 - t)^2/2\varepsilon^2 A$; since $\bar{s}_1 \sim O(1)$, the integrals are only appreciable when $t \sim \varepsilon$. By replacing the Gaussians with a function which is one on $t \in (\varepsilon \bar{s}_1 - \varepsilon, \varepsilon \bar{s}_1 + \varepsilon)$ and zero elsewhere, we obtain

$$\langle I \rangle \sim \int_{\varepsilon \bar{s}_1 - \varepsilon}^{\varepsilon \bar{s}_1 + \varepsilon} dt\, B(t) \tag{67}$$

and

$$\begin{aligned}\langle I^2 \rangle \sim {} & \frac{1}{\varepsilon} \int_{\varepsilon \bar{s}_1 - \varepsilon}^{\varepsilon \bar{s}_1 + \varepsilon} dt[|t| + \tau_c(e^{-|t|/\tau_c} - 1)] \\ & + \frac{1}{\varepsilon} \int_{\varepsilon \bar{s}_1 - \varepsilon}^{\varepsilon \bar{s}_1 + \varepsilon} dt(t - \varepsilon \bar{s}_1) \int_0^t dt'(B(t)\, B(t'))_{\text{sym}}\end{aligned} \tag{68}$$

where $(\cdot)_{\text{sym}}$ is the symmetric product in Eq. (66).

The magnitude of $\langle I \rangle$ depends on the magnitude of B. From the definition of B, Eq. (62), we see that

$$B(t) \sim \frac{1}{\tau_c} \int_0^T dt_1 \, e^{-|t_1 - T - t|/\tau_c} \tag{69}$$

Integration of (69) yields $B(t) \sim O(1)$ for $-T < t < 0$; $B(t)$ decays exponentially to zero when t is outside this interval with decay time τ_c. When $\tau_c \sim O(1)$, $B(t) \sim O(1)$ over the range of integration in (67) and $\langle I \rangle \sim O(\varepsilon)$. When $\tau_c \sim O(\varepsilon)$ or smaller, $B(t)$ may or may not be appreciable in the region of integration. In this case the magnitude of $\langle I \rangle$ depends on the details of $\bar{s}_1$, A, and other $O(1)$ quantities which were previously dropped; we can say, however, that $\langle I \rangle \sim O(\varepsilon)$ or smaller.

The expression (68) for $\langle I^2 \rangle$ contains two terms. Since $t \sim O(\varepsilon)$, the magnitude of the integrand in the first term depends on τ_c:

$$|t| + \tau_c(e^{-|t|/\tau_c} - 1) \sim \begin{cases} |t|^2, & \tau_c \sim O(1) \\ |t|, & \tau_c \sim O(\varepsilon) \text{ or smaller} \end{cases} \tag{70}$$

The first term is therefore $O(\varepsilon^2)$ if $\tau_c \sim O(1)$, and $O(\varepsilon)$ if $\tau_c \sim O(\varepsilon)$ or smaller. From the above discussion of B, the second term is seen to be $O(\varepsilon^2)$ or smaller for all τ_c. Thus, $\langle I^2 \rangle \sim O(\varepsilon^2)$ when $\tau_c \sim O(1)$, and $\langle I^2 \rangle \sim O(\varepsilon)$ when $\tau_c \sim O(\varepsilon)$ or smaller.

When $\tau_c \sim O(1)$, I has an $O(\varepsilon)$ average and rms deviation; hence the magnitude of I is $O(\varepsilon)$. When $\tau_c \sim O(\varepsilon)$ or smaller the rms deviation of I is $O(\varepsilon^{1/2})$ and much larger than the $O(\varepsilon)$ mean; the magnitude of I is now given by its rms deviation. Thus,

$$I \sim \begin{cases} O(\varepsilon), & \tau_c \sim O(1) \\ O(\varepsilon^{1/2}), & \tau_c \sim O(\varepsilon) \text{ or smaller} \end{cases} \tag{71}$$

The magnitudes above are the same as those of an integral similar to I but containing deterministic bounds, $\int_0^\varepsilon dt \, \xi(t)$.

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